

RESOLVENT ESTIMATES OF THE DIRAC OPERATOR

CHRIS PLADDY¹, YOSHIMI SAITŌ², AND TOMIO UMEDA³

^{1,2} Department of Mathematics
University of Alabama at Birmingham
Birmingham, Alabama 35294
U. S. A.

and
³ Department of Mathematics
Himeji Institute of Technology
Himeji 671-22
Japan

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§1. Introduction

The present paper is concerned with the Dirac operator

$$(1.1) \quad H = -i \sum_{j=1}^3 \alpha_j \frac{\partial}{\partial x_j} + \beta + Q(x),$$

where $i = \sqrt{-1}$, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ and α_j, β are the Dirac matrices, i.e., 4×4 Hermitian matrices satisfying the anticommutation relation

$$(1.2) \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \quad (j, k = 1, 2, 3, 4)$$

with the convention $\alpha_4 = \beta$, δ_{jk} being Kronecker's delta and I being the 4×4 identity matrix. The potential $Q(x)$ is a 4×4 Hermitian matrix-valued function, which is usually assumed to diminish at infinity.

The limiting absorption principle for the operator H was first discussed by Yamada [13]. As a result, the existence of the extended resolvents $R^\pm(\lambda)$ was assured (see Theorem 2.2 in section 2 below). The extended resolvents $R^\pm(\lambda)$ play important roles in spectral and scattering theory for the operator H (see [13] and [14]).

The aim of this paper is to investigate the asymptotic behavior of $R^\pm(\lambda)$ as $|\lambda| \rightarrow \infty$. As Yamada [15] pointed out, the operator norm in $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$ of the extended resolvents $R_0^\pm(\lambda)$ of the free Dirac operator

$$(1.3) \quad H_0 = -i \sum_{j=1}^3 \alpha_j \frac{\partial}{\partial x_j} + \beta$$

cannot approach zero as $|\lambda| \rightarrow \infty$. (The definition of $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$ is found below in the introduction.) This means that boundedness of the operator norm of $R_0^\pm(\lambda)$ is possibly the best that one can show. Indeed, one of our main results is that the operator norm of $R_0^\pm(\lambda)$ stay bounded as $|\lambda| \rightarrow \infty$ (see Theorem 2.4 below). However, we also show that $R_0^\pm(\lambda)$ converge strongly to 0 as $|\lambda| \rightarrow \infty$ (see Theorem 2.5). Our results indicate that the extended resolvents of Dirac operators decay much more slowly than those of Schrödinger operators. (Compare Theorems 2.4 – 2.6 with Theorem 4.1.)

We now introduce the notation which will be used in this paper. For $x \in \mathbf{R}^3$, $|x|$ denotes the Euclidean norm of x and

$$(1.4) \quad \langle x \rangle = \sqrt{1 + |x|^2}.$$

For $s \in \mathbf{R}$, we define the weighted Hilbert spaces $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$ by

$$(1.5) \quad L_{2,s}(\mathbf{R}^3) = \{f / \langle x \rangle^s f \in L_2(\mathbf{R}^3)\},$$

and

$$(1.6) \quad H_s^1(\mathbf{R}^3) = \{f / \langle x \rangle^s \partial_x^\alpha f \in L_2(\mathbf{R}^3), |\alpha| \leq 1\},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and

$$(1.7) \quad \left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

The inner products and norms in $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$ are given by

$$(1.8) \quad \begin{cases} (f, g)_s = \int_{\mathbf{R}^3} \langle x \rangle^{2s} f(x) \overline{g(x)} dx, \\ \|f\|_s = [(f, f)_s]^{1/2}, \end{cases}$$

and

$$(1.9) \quad \begin{cases} (f, g)_{1,s} = \int_{\mathbf{R}^3} \langle x \rangle^{2s} [f(x) \overline{g(x)} + \nabla f(x) \cdot \overline{\nabla g(x)}] dx, \\ \|f\|_{1,s} = [(f, f)_{1,s}]^{1/2}, \end{cases}$$

respectively. The spaces $\mathcal{L}_{2,s}$ and \mathcal{H}_s^1 are defined by

$$(1.10) \quad \begin{cases} \mathcal{L}_{2,s} = [L_{2,s}(\mathbf{R}^3)]^4, \\ \mathcal{H}_s^1 = [H_s^1(\mathbf{R}^3)]^4, \end{cases}$$

i.e., $\mathcal{L}_{2,s}$ and \mathcal{H}_s^1 are direct sums of the Hilbert spaces $L_{2,s}(\mathbf{R}^3)$ and $H_s^1(\mathbf{R}^3)$, respectively. The inner products and norms in $\mathcal{L}_{2,s}$ and \mathcal{H}_s^1 are also denoted by $(\cdot, \cdot)_s$, $\|\cdot\|_s$ and $(\cdot, \cdot)_{1,s}$, $\|\cdot\|_{1,s}$, respectively. When $s = 0$, we simply write

$$(1.11) \quad \begin{cases} \mathcal{L}_2 = \mathcal{L}_{2,0}, \\ \mathcal{H}^1 = \mathcal{H}_0^1. \end{cases}$$

For a pair of Hilbert spaces X and Y , $\mathbf{B}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y , equipped with the operator norm

$$(1.12) \quad \|T\| = \sup_{x \in X \setminus \{0\}} \|Tx\|_Y / \|x\|_X,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms in X and Y . For $T \in \mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,t})$, its operator norm will be denoted by $\|T\|_{(s,t)}$.

We now sketch the contents of the paper. In section 2, we state the main theorems. For the reader's convenience, we reproduce Yamada's arguments [15] in section 3. In section 4, we make a brief review of resolvent estimates for Schrödinger operators which will be used in the proof of Theorem 2.4. In section 5, we establish some boundedness results for pseudodifferential operators acting in the weighted Hilbert spaces, the results on which the proof of Theorem 2.4 is based. We give the proofs of Theorems 2.4 and 2.5 in sections 6 and 7 respectively. In section 8, we give the proof of Theorem 2.6.

Finally, we would like to mention that Pladdy, Saitō and Umeda [6] is an announcement for this work. Also, We would like to mention that we can establish resolvent estimates for relativistic Schrödinger operators $\sqrt{-\Delta + m^2} + V(x)$, the estimates which are similar to those of the Dirac operators. Discussions about the resolvent estimates for the relativistic Schrödinger operators will appear elsewhere.

The present work was done while the last author (T.U.) was visiting the Department of Mathematics, the University of Alabama at Birmingham for the 1992–93 academic year. He would like to express his sincere gratitude to the members of the department for their warm hospitality. He also would like to thank Himeji Institute of Technology for allowing him to take a year's leave of absence.

§2. Main results

We begin with the selfadjointness of the free Dirac operator H_0 . It is known that H_0 restricted on $[C_0^\infty(\mathbf{R}^3)]^4$ is essentially selfadjoint in \mathcal{L}_2 and its selfadjoint extension, which will be denoted by H_0 again, has the domain \mathcal{H}^1 .

We impose the following assumption on the potential.

Assumption 2.1.

- (i) $Q(x) = (q_{jk}(x))$ is a 4×4 Hermitian matrix-valued C^1 function on \mathbf{R}^3 ;
- (ii) There exist positive constants ϵ and K such that

$$(2.1) \quad \langle x \rangle^{1+\epsilon} |q_{jk}(x)| + \sum_{\ell=1}^3 \left| \frac{\partial q_{jk}}{\partial x_\ell}(x) \right| \leq K$$

for $j, k = 1, 2, 3, 4$.

Assumption 2.1 is essentially the same one as Yamada made in [13]. He needs the first derivatives of q_{jk} bounded in order to assure that the Dirac operator

H has no embedded eigenvalues in its essential spectrum; see [13, Proposition 2.5].

Under Assumption 2.1 the multiplication operator $Q = Q(x) \times$ is a bounded selfadjoint operator in \mathcal{L}_2 . Hence, by the Kato-Rellich theorem (Kato[4], p.287), H restricted on $[C_0^\infty(\mathbf{R}^3)]^4$ is also essentially selfadjoint in \mathcal{L}_2 and its selfadjoint extension, which will be denoted by H again, has the same domain \mathcal{H}^1 as H_0 . We write

$$(2.2) \quad R_0(z) = (H_0 - z)^{-1},$$

and

$$(2.3) \quad R(z) = (H - z)^{-1}.$$

As we mentioned in the introduction, the limiting absorption principle holds for the Dirac operator H . We note that in Theorem 2.2 below $R(z)$ is regarded as an operator belonging to $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$.

Theorem 2.2 (Yamada [13]).

Suppose that Assumption 2.1 is satisfied and let $s > 1/2$. Then for any $\lambda \in (-\infty, -1) \cup (1, \infty)$, there exist the extended resolvents $R^\pm(\lambda) \in \mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$ such that for any $f \in \mathcal{L}_{2,s}$

$$R(\lambda \pm i\mu)f \longrightarrow R^\pm(\lambda)f \quad \text{in } \mathcal{L}_{2,-s}$$

as $\mu \downarrow 0$. Moreover, for $f \in \mathcal{L}_{2,s}$, $R^\pm(\lambda)f$ is an $\mathcal{L}_{2,-s}$ -valued continuous function on $(-\infty, -1) \cup (1, \infty)$.

Remark 2.3.

- (i) Actually, Yamada [13] proved that $R^\pm(\lambda)$ belong to $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{H}_{-s}^1)$. In particular, $R^\pm(\lambda)$ belong to $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,-s})$, which is suitable for our purpose.
- (ii) Note that the conclusions of Theorem 2.2 are valid, in particular, for the resolvent of the free Dirac operator H_0 .

We now state the main theorems, which are concerned with the asymptotic behavior of the extended resolvents $R_0^\pm(\lambda)$ of the free Dirac operator H_0 .

Theorem 2.4.

Let $s > 1/2$. Then

$$\|R_0^\pm(\lambda)\|_{(s,-s)} = O(1) \quad (|\lambda| \rightarrow \infty).$$

As we shall see later, $\|R_0^\pm(\lambda)\|_{(s,-s)}$ cannot be small no matter how $|\lambda|$ is large. In this sense the estimate in Theorem 2.4 is best possible. However, $R_0^\pm(\lambda)$ do become small in strong operator topology as $|\lambda|$ gets large. In fact, we have

Theorem 2.5.

Let $s > 1/2$. Then $R_0^\pm(\lambda)$ converge strongly to 0 as $|\lambda| \rightarrow \infty$, i.e., for any $f \in \mathcal{L}_{2,s}$

$$(2.4) \quad R_0^\pm(\lambda)f \longrightarrow 0 \quad \text{in } \mathcal{L}_{2,-s}$$

as $|\lambda| \rightarrow \infty$.

Based on Theorems 2.4 and 2.5, the Dirac operator with a small coupling constant can be handled; we can use the Neumann series expansion. Let

$$H_t = -i \sum_{j=1}^3 \alpha_j \frac{\partial}{\partial x_j} + \beta + t Q(x),$$

where t is a real number. The extended resolvents of H_t will be denoted by $R_t^\pm(\lambda)$. Then we have the following

Theorem 2.6.

Suppose that $Q(x)$ satisfies Assumption 2.1 and let $s > 1/2$. Then there exist constants $t_0 > 0$ and $C > 0$ such that for every t with $|t| \leq t_0$

- (i) $\sup_{|\lambda| \geq 2} \|R_t^\pm(\lambda)\|_{(s,-s)} \leq C$,
- (ii) $R_t^\pm(\lambda)$ converge strongly to 0 as $|\lambda| \rightarrow \infty$.

§3. Yamada's counterexample

In this section, we shall reproduce Yamada's arguments [15] to show that $\|R_0^\pm(\lambda)\|_{(s,-s)}$ cannot converge to 0 as $|\lambda| \rightarrow \infty$. In Proposition 3.1 below and in the rest of the paper, $\mathcal{S}(\mathbf{R}^3)$ denotes the Schwartz space of rapidly decreasing functions on \mathbf{R}^3 .

Proposition 3.1 (Yamada [15]).

- There exists a sequence $\{h_n\}_{n=1}^\infty \subset [\mathcal{S}(\mathbf{R}^3)]^4$ such that*
- (i) $\sup_n \|h_n\|_s < +\infty$ *for every $s > 0$,*
 - (ii) $\lim_{n \rightarrow \infty} (R_0^\pm(n+2)h_n, h_n)_0 \neq 0$.

It follows from Proposition 3.1 that for any $s > 1/2$, $\|R_0^\pm(\lambda)\|_{(s,-s)}$ cannot converge to 0 as $\lambda \rightarrow \infty$. In fact, the inequality

$$|(R_0^\pm(n+2)h_n, h_n)_0| \leq \|R_0^\pm(n+2)\|_{(s,-s)} \|h_n\|_s^2,$$

together with Proposition 3.1, implies that

$$\liminf_{n \rightarrow \infty} \|R_0^\pm(n+2)\|_{(s,-s)} > 0.$$

Remark 3.2.

One can also show that for any $s > 1/2$, $\|R_0^\pm(\lambda)\|_{(s,-s)}$ cannot converge to 0 as $\lambda \rightarrow -\infty$. See Yamada [15].

Throughout (and only in) this section, we assume that

$$(3.1) \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This causes no loss of generality. Indeed, H_0 with any Dirac matrices is unitarily equivalent to H_0 with the Dirac matrices α_j and β of the form (3.1).

We will give the proof of Proposition 3.1 with a series of lemmas.

Lemma 3.3.

Let φ be a real-valued C^1 -function defined on $[-1, 1]$. Then

$$\lim_{\mu \downarrow 0} \int_{-1}^1 \frac{\varphi(\sigma)}{\sigma \mp i\mu} d\sigma = \pm i\pi\varphi(0) + \int_{-1}^1 \left\{ \int_0^1 \varphi'(\sigma\theta) d\theta \right\} d\sigma.$$

Proof. It is easy to see that

$$(3.2) \quad \int_{-1}^1 \frac{\varphi(\sigma)}{\sigma \mp i\mu} d\sigma = \varphi(0) \int_{-1}^1 \frac{1}{\sigma \mp i\mu} d\sigma + \int_{-1}^1 \frac{\varphi(\sigma) - \varphi(0)}{\sigma \mp i\mu} d\sigma.$$

Noting that

$$\varphi(\sigma) - \varphi(0) = \sigma \int_0^1 \varphi'(\sigma\theta) d\theta,$$

and taking the limit of (3.2) as $\mu \downarrow 0$, we get the desired conclusion. \square

Lemma 3.4.

Let φ be a real-valued C^1 -function defined on $[-1, 1]$ and suppose that φ is an even function. Then

$$\int_{-1}^1 \left\{ \int_0^1 \varphi'(\sigma\theta) d\theta \right\} d\sigma = 0.$$

Proof. Since φ is an even function, we see that φ' is an odd function. Then $\int_0^1 \varphi'(\sigma\theta) d\theta$ is also an odd function of σ , of which integral from -1 to 1 is equal to 0 . \square

To prove Proposition 3.1, we shall construct the sequence $\{h_n\}$ in the following manner: First choose an even function $\varphi \in C_0^\infty(\mathbf{R})$ so that

$$(3.3) \quad \text{supp}[\varphi] \subset (-1, 1)$$

and

$$(3.4) \quad \varphi(0) = 1.$$

Next, define $a_n \in \mathcal{S}(\mathbf{R}^3)$ by

$$(3.5) \quad \widehat{a}_n(\xi) = \frac{1}{|\xi|} \varphi(\langle \xi \rangle - n - 2) \quad (n = 1, 2, \dots),$$

where $\widehat{a} = \mathcal{F}a$ is the Fourier transform of a :

$$\widehat{a}(\xi) = [\mathcal{F}a](\xi) = \int_{\mathbf{R}^3} e^{-ix \cdot \xi} a(x) dx.$$

Later we will also use the inverse Fourier transform which is given by

$$[\mathcal{F}^{-1}b](x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix \cdot \xi} b(\xi) d\xi.$$

Note that

$$(3.6) \quad \text{supp}[\widehat{a}_n] \subset \{ \xi \in \mathbf{R}^3 / \sqrt{n(n+2)} \leq |\xi| \leq \sqrt{n^2 + 6n + 8} \}.$$

Finally, define $h_n \in [\mathcal{S}(\mathbf{R}^3)]^4$ by

$$(3.7) \quad h_n = \begin{pmatrix} a_n \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (n = 1, 2, \dots).$$

Lemma 3.5.

For any $s > 0$

$$\sup_n \|h_n\|_s < +\infty.$$

Proof. In view of (3.7), it is sufficient to show that for any multi-index α

$$\sup_n \|x^\alpha a_n\|_0 < +\infty.$$

By integration by parts, we see that

$$(3.8) \quad x^\alpha a_n(x) = (2\pi)^{-3} \int e^{ix \cdot \xi} \left(i \frac{\partial}{\partial \xi}\right)^\alpha \widehat{a}_n(\xi) d\xi.$$

Combining (3.8) with (3.5), (3.6) and using the Plancherel theorem, we get

$$\|x^\alpha a_n\|_0^2 \leq C_{\alpha\varphi} \int_{\sqrt{n(n+2)} \leq |\xi| \leq \sqrt{n^2+6n+8}} |\xi|^{-2} d\xi,$$

where the constant $C_{\alpha\varphi}$ depends only on α and the least upper bound of φ , together with its all derivatives up to $|\alpha|$ -th order. This gives the desired conclusion. \square

We note that the resolvent $R_0(z)$ of the free Dirac operator H_0 can be represented in terms of the Fourier transform:

$$(3.9) \quad R_0(z) = \mathcal{F}^{-1} [(\widehat{L}_0(\xi) - zI)^{-1}] \mathcal{F} \quad (\text{Im } z \neq 0)$$

where

$$(3.10) \quad \widehat{L}_0(\xi) = \sum_{j=1}^3 \xi_j \alpha_j + \beta.$$

Here an explanation must be needed. We define the Fourier transform of a \mathbf{C}^4 -valued function

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}$$

by

$$\widehat{f}(\xi) = [\mathcal{F}f](\xi) = \begin{pmatrix} \widehat{f}_1(\xi) \\ \widehat{f}_2(\xi) \\ \widehat{f}_3(\xi) \\ \widehat{f}_4(\xi) \end{pmatrix}.$$

For every $\xi \in \mathbf{R}^3$, the Hermitian matrix $\widehat{L}_0(\xi)$, acting in \mathbf{C}^4 with the usual inner product, has two eigenvalues $\pm\langle\xi\rangle$, each of which is an eigenvalue of multiplicity two. The property

$$(3.11) \quad (\widehat{L}_0(\xi))^2 = \langle\xi\rangle^2 I$$

implies that the eigenprojections $\Psi_{\pm}(\xi)$ associated with the eigenvalues $\pm\langle\xi\rangle$ of $\widehat{L}_0(\xi)$ are given by

$$(3.12) \quad \Psi_{\pm}(\xi) = \frac{1}{2} \left(I \pm \frac{1}{\langle\xi\rangle} \widehat{L}_0(\xi) \right)$$

respectively (cf. [14, §1]). Therefore

$$(3.13) \quad R_0(z) = \mathcal{F}^{-1} \left[-\frac{1}{\langle\xi\rangle + z} \Psi_{-}(\xi) + \frac{1}{\langle\xi\rangle - z} \Psi_{+}(\xi) \right] \mathcal{F}.$$

Lemma 3.6.

Let $\{h_n\}$ be the sequence given by (3.7). Then

$$\lim_{n \rightarrow \infty} (R_0^{\pm}(n+2)h_n, h_n)_0 = \pm \frac{i}{4\pi}.$$

Proof. To simplify the notation, we give the proof only for “+”. Since, by (3.5), $\widehat{h}_n(\xi)$ is an even function of ξ_j and $\xi_j \widehat{h}_n(\xi)$ is an odd function of ξ_j , we see that

$$(3.14) \quad \int \frac{1}{\langle\xi\rangle \pm z} \left\langle \frac{1}{\langle\xi\rangle} \xi_j \alpha_j \widehat{h}_n(\xi), \widehat{h}_n(\xi) \right\rangle d\xi = 0 \quad (j = 1, 2, 3)$$

where α_j is the matrix given in (1.1) and

$$\langle \widehat{f}(\xi), \widehat{g}(\xi) \rangle = \sum_{k=1}^4 \widehat{f}_k(\xi) \overline{\widehat{g}_k(\xi)}.$$

Taking into account (3.10), (3.12)—(3.14) and (3.7), we get

$$\begin{aligned} (R_0(z)h_n, h_n)_0 &= (2\pi)^{-3} \int \frac{-1}{\langle\xi\rangle + z} \cdot \frac{1}{2} \left(1 - \frac{1}{\langle\xi\rangle}\right) |\widehat{a}_n(\xi)|^2 d\xi \\ &\quad + (2\pi)^{-3} \int \frac{1}{\langle\xi\rangle - z} \cdot \frac{1}{2} \left(1 + \frac{1}{\langle\xi\rangle}\right) |\widehat{a}_n(\xi)|^2 d\xi, \end{aligned}$$

where we used (3.1). Using (3.5) and passing to the polar coordinates, we have

$$(3.15) \quad \left\{ \begin{aligned} & 4\pi^2 (R_0(z)h_n, h_n)_0 \\ &= - \int_0^\infty \frac{1}{\sqrt{r^2+1}+z} \left(1 - \frac{1}{\sqrt{r^2+1}}\right) \varphi(\sqrt{r^2+1}-n-2)^2 dr \\ & \quad + \int_0^\infty \frac{1}{\sqrt{r^2+1}-z} \left(1 + \frac{1}{\sqrt{r^2+1}}\right) \varphi(\sqrt{r^2+1}-n-2)^2 dr \\ &= - \int_{-1}^1 \frac{1}{\sigma+n+2+z} \left(1 - \frac{1}{\sigma+n+2}\right) \varphi(\sigma)^2 \frac{\sigma+n+2}{\sqrt{(\sigma+n+2)^2-1}} d\sigma \\ & \quad + \int_{-1}^1 \frac{1}{\sigma+n+2-z} \left(1 + \frac{1}{\sigma+n+2}\right) \varphi(\sigma)^2 \frac{\sigma+n+2}{\sqrt{(\sigma+n+2)^2-1}} d\sigma. \end{aligned} \right.$$

In the second equality above, we made a change of a variable. Putting

$$z = n+2+i\mu \quad (\mu > 0),$$

and taking the limit of (3.15) as $\mu \downarrow 0$, we see, in view of Theorem 2.2 and Lemma 3.3, that

$$(3.16) \quad \left\{ \begin{aligned} & 4\pi^2 (R_0^+(n+2)h_n, h_n)_0 \\ &= - \int_{-1}^1 \frac{1}{\sigma+2n+4} \left(1 - \frac{1}{\sigma+n+2}\right) \varphi(\sigma)^2 \frac{\sigma+n+2}{\sqrt{(\sigma+n+2)^2-1}} d\sigma \\ & \quad + i\pi \left(1 + \frac{1}{n+2}\right) \varphi(0)^2 \frac{n+2}{\sqrt{(n+2)^2-1}} \\ & \quad + \int_{-1}^1 \left\{ \int_0^1 \omega'_n(\sigma\theta) d\theta \right\} d\sigma, \end{aligned} \right.$$

where

$$\omega_n(\sigma) = \left(1 + \frac{1}{\sigma+n+2}\right) \varphi(\sigma)^2 \frac{\sigma+n+2}{\sqrt{(\sigma+n+2)^2-1}}.$$

It is easy to see that the integrand in the first term on the right hand side of (3.16) is less than or equal to, in the absolute value, K/n where K is a positive constant independent of n . Therefore, the first term converges to 0 as $n \rightarrow \infty$. As for the third term on the right hand side of (3.16), we see that $\{\omega'_n\}_{n=1}^\infty$ is a uniformly bounded sequence of functions which converges pointwisely to $(\varphi^2)'$. Hence, by the Lebesgue dominated convergence theorem,

$$(3.17) \quad \lim_{n \rightarrow \infty} \int_{-1}^1 \left\{ \int_0^1 \omega'_n(\sigma\theta) d\theta \right\} d\sigma = \int_{-1}^1 \left\{ \int_0^1 (\varphi^2)'(\sigma\theta) d\theta \right\} d\sigma.$$

In view of the fact that φ^2 is an even function, it follows from Lemma 3.4 that the right hand side of (3.17) equals 0. Summing up, we get

$$\lim_{n \rightarrow \infty} 4\pi^2 (R_0^+(n+2)h_n, h_n)_0 = i\pi \varphi(0)^2 = i\pi.$$

This completes the proof. \square

It is obvious that Lemmas 3.5 and 3.6 give the proof of Proposition 3.1.

§4. A known result for Schrödinger operators

The limiting absorption principle for Schrödinger operators has been extensively studied in connection with the spectral and scattering theory; cf. [8], [3], [1]. In this section, we make a brief review of a result due to Saitō [9] and [10], which will be used in the proof of Theorem 2.4. Let T denote the selfadjoint operator which is defined to be the closure of $-\Delta + V(x)$ restricted on $C_0^\infty(\mathbf{R}^n)$, where $V(x)$ is a real-valued function satisfying

$$(4.1) \quad |V(x)| \leq C \langle x \rangle^{-1-\epsilon}$$

for $C > 0$ and $\epsilon > 0$. Let

$$\Gamma(z) = (T - z)^{-1}.$$

Then it is well-known that the limiting absorption principle holds for T , that is, for any $\lambda > 0$, there correspond the extended resolvents $\Gamma^\pm(\lambda)$ in $\mathbf{B}(L_{2,s}(\mathbf{R}^n), L_{2,-s}(\mathbf{R}^n))$ such that for any f in $L_{2,s}(\mathbf{R}^n)$

$$(4.2) \quad \Gamma(\lambda \pm i\mu)f \longrightarrow \Gamma^\pm(\lambda)f \quad \text{in } L_{2,-s}$$

as $\mu \downarrow 0$. Furthermore, it is known that $\Gamma^\pm(\lambda)f$ are $L_{2,-s}(\mathbf{R}^n)$ -valued continuous functions in $(0, \infty)$. (Saitō [8], Ikebe-Saitō [3] and Agmon [1].) As for asymptotic behaviors of $\Gamma^\pm(\lambda)$, we have

Theorem 4.1 (Saitō [9, 10]).

$$(4.3) \quad \|\Gamma^\pm(\lambda)\|_{(s,-s)} = O(\lambda^{-1/2}) \quad (\lambda \rightarrow \infty).$$

Also, Saitō proved

Theorem 4.2 (Saitō [9, 10]).

Let $s > 1/2$. Then for any $d > 0$ there exists a positive constant $C > 0$ such that

$$(4.4) \quad \|\Gamma(\kappa^2)\|_{(s, -s)} \leq C/|\kappa|$$

for all κ with $|\operatorname{Re} \kappa| > d$ and $\operatorname{Im} \kappa > 0$.

Remark 4.3

We would like to emphasize that the asymptotic behavior (4.3) is useful in the inverse scattering theory for the Schrödinger operator. See Saitō [11, 12].

§5. Pseudodifferential operators

The proof of Theorem 2.4 is based on the resolvent estimate for the Schrödinger operator (Theorem 4.2) as well as the theory of pseudodifferential operators. In this section we introduce a class of symbols of pseudo-differential operators which are suitable to our purpose. We then establish boundedness results in the weighted Hilbert spaces, which are important in relation to the limiting absorption principle for the Dirac operator; cf. [7].

Definition 5.1.

A C^∞ function $p(x, \xi)$ on $\mathbf{R}^3 \times \mathbf{R}^3$ is said to be in the class $S_{0,0}^m$ ($m \in \mathbf{R}$) if for any pair α and β of multi-indices there exists a constant $C_{\alpha\beta} \geq 0$ such that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta p(x, \xi) \right| \leq C_{\alpha\beta} \langle \xi \rangle^m$$

for all $x, \xi \in \mathbf{R}^3$.

Remark 5.2.

The class $S_{0,0}^m$ is a Fréchet space equipped with the semi-norms

$$|p|_\ell^{(m)} = \max_{|\alpha|, |\beta| \leq \ell} \sup_{x, \xi} \left\{ \langle \xi \rangle^{-m} \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta p(x, \xi) \right| \right\} \quad (\ell = 0, 1, 2, \dots).$$

A pseudodifferential operator $p(x, D)$ with symbol $p(x, \xi)$ is defined by

$$p(x, D)f(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix \cdot \xi} p(x, \xi) \widehat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbf{R}^3)$.

Lemma 5.3.

Let $p(x, \xi)$ be in $S_{0,0}^0$ and let $s > 0$. Define the oscillatory integral

$$r(x, \xi) = \text{Os} - \iint_{\mathbf{R}^6} e^{-iy \cdot \eta} p(x, \xi + \eta) \langle x + y \rangle^{-s} (2\pi)^{-3} dy d\eta.$$

Then for any pair α and β there exists a constant $C_{s\alpha\beta} \geq 0$ such that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta r(x, \xi) \right| \leq C_{s\alpha\beta} |p|_k^{(0)} \langle x \rangle^{-s},$$

where

$$(5.1) \quad k = \max \left\{ |\beta|, 2[s + \frac{5}{2}] + |\alpha| \right\}.$$

Remark 5.4.

- (i) For the definition of oscillatory integral, see Kumano-go[5, Chapter 1, Section 6].
- (ii) For a positive number s , $[s]$ denotes the largest integer less than or equal to s .
- (iii) By [5, Theorem 2.6(1), p. 74], $r(x, D) = p(x, D) \langle x \rangle^{-s}$. This fact will be used in the proof of Lemma 5.5 below.

Proof. By differentiation under the oscillatory integral sign (cf. [5, (2.23), p. 70]), we see that

$$(5.2) \quad \begin{aligned} & \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta r(x, \xi) \\ &= \text{Os} - \iint_{\mathbf{R}^6} e^{-iy \cdot \eta} \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta \{ p(x, \xi + \eta) \langle x + y \rangle^{-s} \} (2\pi)^{-3} dy d\eta. \end{aligned}$$

Putting

$$M = \left[\frac{s+5}{2} \right],$$

and integrating by parts, we see that

$$(5.3) \quad \begin{aligned} \text{RHS of (5.2)} &= \iint_{\mathbf{R}^6} e^{-iy \cdot \eta} \langle y \rangle^{-2M} \langle D_\eta \rangle^{2M} \{ \langle \eta \rangle^{-4} \langle D_y \rangle^4 \times \\ &\quad \times \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta \{ p(x, \xi + \eta) \langle x + y \rangle^{-s} \} \} (2\pi)^{-3} dy d\eta, \end{aligned}$$

where

$$\langle D_\eta \rangle^2 = 1 - \Delta_\eta; \quad \langle D_y \rangle^2 = 1 - \Delta_y$$

(cf. [5, Theorem 6.4, p. 47]). Note that the integral on the right hand side of (5.3) is in the usual sense. Using the following two inequalities

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \langle x \rangle^{-s} \right| \leq C_{\alpha s} \langle x \rangle^{-s}$$

and

$$\langle x + y \rangle^{-1} \leq \sqrt{2} \langle y \rangle \langle x \rangle^{-1},$$

we get

$$(5.4) \quad \begin{aligned} & | \text{the integrand on the RHS of (5.3)} | \\ & \leq C_{s\alpha\beta} |p|_k^{(0)} \langle x \rangle^{-s} \langle y \rangle^{-2M+s} \langle \eta \rangle^{-4}, \end{aligned}$$

where $C_{s\alpha\beta}$ is a nonnegative constant. We note that $\langle y \rangle^{-2M+s} \langle \eta \rangle^{-4}$ is integrable on \mathbf{R}^6 . Hence, combining (5.2)–(5.4), we get the desired conclusion. \square

Lemma 5.5

Let $p(x, \xi)$ be in $S_{0,0}^0$. Then for any $s \geq 0$ there exist a nonnegative constant C and a positive integer ℓ such that

$$(5.5) \quad \|p(x, D)f\|_s \leq C |p|_\ell^{(0)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^3)),$$

where C and ℓ depend only on s .

Proof. It is sufficient to show that for any $s > 0$ there exist a nonnegative constant C and a positive integer ℓ such that

$$(5.6) \quad \|\langle x \rangle^s p(x, D) \langle x \rangle^{-s} f\|_0 \leq C |p|_\ell^{(0)} \|f\|_0$$

for all $f \in \mathcal{S}(\mathbf{R}^3)$. Let $r(x, \xi)$ be the symbol defined in Lemma 5.3, and put

$$q(x, \xi) = \langle x \rangle^s r(x, \xi).$$

According to Remark 5.4(iii),

$$q(x, D) = \langle x \rangle^s p(x, D) \langle x \rangle^{-s}.$$

It follows from Lemma 5.3 that for any pair α and β there exists a constant $C_{s\alpha\beta} \geq 0$ such that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left(\frac{\partial}{\partial x} \right)^\beta q(x, \xi) \right| \leq C_{s\alpha\beta} |p|_k^{(0)}$$

where k is given by (5.1). Then the Calderón-Vaillancourt theorem([2], [5, Theorem 1.6, p.224]) implies (5.6). \square

Lemma 5.6.

Let $p(\xi)$ be in $S_{0,0}^{-1}$. Then for any $s \geq 0$ there exist a nonnegative constant C and a positive integer ℓ such that

$$(5.7) \quad \|p(D)f\|_{1,s} \leq C |p|_{\ell}^{(-1)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^3)),$$

where C and ℓ depend only on s .

Proof. By definition (1.9), we have

$$(5.8) \quad \|p(D)f\|_{1,s}^2 = \|p(D)f\|_s^2 + \sum_{j=1}^3 \left\| \frac{\partial}{\partial x_j} p(D)f \right\|_s^2.$$

Regarding $p(\xi)$ as a symbol in $S_{0,0}^0$, and applying Lemma 5.5, we get

$$(5.9) \quad \|p(D)f\|_s \leq C |p|_{\ell}^{(0)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^3)).$$

Note that the symbol of $(\partial/\partial x_j)p(D)$ is $i\xi_j p(\xi)$, which belongs to $S_{0,0}^0$. Then, by Lemma 5.5, we see that

$$(5.10) \quad \left\| \frac{\partial}{\partial x_j} p(D)f \right\|_s \leq C |p|_{\ell}^{(-1)} \|f\|_s \quad (f \in \mathcal{S}(\mathbf{R}^3)).$$

Using the fact that $|p|_{\ell}^{(0)} \leq |p|_{\ell}^{(-1)}$ for $\ell = 0, 1, 2, \dots$, and combining (5.8)–(5.10), we obtain (5.7). \square

We now need to extend Lemmas 5.5 and 5.6 to a system of pseudodifferential operators. Let

$$P(x, \xi) = (p_{jk}(x, \xi))_{1 \leq j, k \leq 4}$$

be a 4×4 matrix-valued symbol. Then we define

$$P(x, D) = (p_{jk}(x, D))_{1 \leq j, k \leq 4}$$

by

$$P(x, D)f(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix \cdot \xi} P(x, \xi) \widehat{f}(\xi) d\xi$$

for $f \in [\mathcal{S}(\mathbf{R}^3)]^4$. If $p_{jk}(x, \xi) \in S_{0,0}^m$, $1 \leq j, k \leq 4$, we define

$$(5.11) \quad |P|_{\ell}^{(m)} = \left\{ \sum_{j,k=1}^4 (|p_{jk}|_{\ell}^{(m)})^2 \right\}^{1/2}$$

for $\ell = 0, 1, 2, \dots$, where $|p_{jk}|_{\ell}^{(m)}$ are the semi-norms introduced in Remark 5.2. We then have natural extensions of Lemmas 5.5 and 5.6.

Lemma 5.7.

Let $p_{jk}(x, \xi)$ be in $S_{0,0}^0$ for $j, k = 1, 2, 3, 4$. Then for any $s \geq 0$ there exist a nonnegative constant C and a positive integer ℓ such that

$$(5.12) \quad \|P(x, D)f\|_s \leq C |P|_\ell^{(0)} \|f\|_s \quad (f \in [\mathcal{S}(\mathbf{R}^3)]^4),$$

where C and ℓ depend only on s .

Proof. It is a matter of simple computation:

$$\begin{aligned} \|P(x, D)f\|_s^2 &= \sum_{j=1}^4 \left\| \sum_{k=1}^4 p_{jk}(x, D) f_k \right\|_s^2 \\ &\leq \sum_{j=1}^4 \left(\sum_{k=1}^4 C |p_{jk}|_\ell^{(0)} \|f_k\|_s \right)^2 \quad (\text{by Lemma 5.5}) \\ &\leq C^2 \sum_{j=1}^4 \left\{ \sum_{k=1}^4 \left(|p_{jk}|_\ell^{(0)} \right)^2 \right\} \sum_{k=1}^4 \|f_k\|_s^2 \quad (\text{by the Schwarz inequality}). \end{aligned}$$

With the notation (5.11), this is equivalent to (5.12). \square

Lemma 5.8.

Let $p_{jk}(\xi)$ be in $S_{0,0}^{-1}$ for $j, k = 1, 2, 3, 4$. Then for any $s \geq 0$ there exist a nonnegative constant C and a positive integer ℓ such that

$$\|P(D)f\|_{1,s} \leq C |P|_\ell^{(-1)} \|f\|_s \quad (f \in [\mathcal{S}(\mathbf{R}^3)]^4),$$

where C and ℓ depend only on s .

In view of Lemma 5.6, the proof of Lemma 5.8 is similar to that of Lemma 5.7. We should like to mention that Lemma 5.8 is beyond the necessity for the present paper. However, we need the lemma in our forthcoming paper [7].

§6. Proof of Theorem 2.4

In this section, we give the proof of Theorem 2.4. We begin with rewriting (3.9). Using (3.11), we see that

$$(6.1) \quad (\widehat{L}_0(\xi) - zI) (\widehat{L}_0(\xi) + zI) = (\langle \xi \rangle^2 - z^2) I.$$

Hence

$$(6.2) \quad R_0(z) = \mathcal{F}^{-1} \left[\frac{1}{\langle \xi \rangle^2 - z^2} (\widehat{L}_0(\xi) + zI) \right] \mathcal{F} \quad (\text{Im } z \neq 0).$$

Theorem 6.1.

Suppose that $s > 1/2$. Then

$$(6.3) \quad \sup \left\{ \|R_0(z)\|_{(s,-s)} \mid 2 \leq |\text{Re } z|, \ 0 < |\text{Im } z| < 1 \right\} < +\infty.$$

Remark 6.2.

It is evident that Theorem 6.1, together with Theorem 2.2, implies Theorem 2.4.

Proof. Set

$$J = \left\{ z \in \mathbf{C} \mid 2 \leq |\text{Re } z|, \ 0 < |\text{Im } z| < 1 \right\}.$$

Choose $\rho \in C_0^\infty(\mathbf{R})$ so that

$$\rho(t) = \begin{cases} 1, & \text{if } |t| < 1/2 \\ 0, & \text{if } |t| > 1. \end{cases}$$

For each $z \in J$, we define a cutoff function $\gamma_z(\xi)$ on \mathbf{R}^3 by

$$\gamma_z(\xi) = \begin{cases} \rho(\langle \xi \rangle - \text{Re } z), & \text{if } \text{Re } z \geq 2 \\ \rho(\langle \xi \rangle + \text{Re } z), & \text{if } \text{Re } z \leq -2. \end{cases}$$

Using (6.2) and $\gamma_z(\xi)$, we decompose the resolvent of H_0 into three parts:

$$R_0(z) = (-\Delta + 1 - z^2)^{-1} A_z + B_z + z(-\Delta + 1 - z^2)^{-1}$$

where

$$\begin{aligned} A_z &= \mathcal{F}^{-1} \left[\gamma_z(\xi) \widehat{L}_0(\xi) \right] \mathcal{F}, \\ B_z &= \mathcal{F}^{-1} \left[\frac{1 - \gamma_z(\xi)}{\langle \xi \rangle^2 - z^2} \widehat{L}_0(\xi) \right] \mathcal{F}. \end{aligned}$$

Note that for $\xi \in \text{supp}[\gamma_z]$ with $z \in J$

$$(6.4) \quad \frac{1}{4}|z| \leq \langle \xi \rangle \leq \frac{3}{2}|z|.$$

Using (6.4) and (3.10), we see that for any α there exists a constant C_α such that

$$(6.5) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha (\gamma_z(\xi) \widehat{L}_0(\xi)) \right| \leq C_\alpha |z|$$

for all $z \in J$. Here and in the sequel, for a 4×4 matrix M , its matrix norm is denoted by $|M|$ (e.g., $|M|^2 = \sum_{j,k=1}^4 m_{jk}^2$; actually it is irrelevant which norm one chooses), and for a 4×4 matrix-valued function $M(\xi) = (m_{jk}(\xi))$ we write

$$\left(\frac{\partial}{\partial \xi} \right)^\alpha M(\xi) = \left(\left(\frac{\partial}{\partial \xi} \right)^\alpha m_{jk}(\xi) \right)_{1 \leq j, k \leq 4}.$$

Then noting (6.5) and applying Lemma 5.7 to A_z , we get

$$(6.6) \quad \|A_z f\|_s \leq C_1 |z| \|f\|_s \quad (f \in [\mathcal{S}(\mathbf{R}^3)]^4)$$

for $z \in J$, where C_1 is independent of $z \in J$. On the other hand, it follows, in particular, from Theorem 4.2 that for $z \in J$

$$(6.7) \quad \|(-\Delta + 1 - z^2)^{-1}\|_{(s, -s)} \leq \frac{C_2}{|z|}$$

with a constant C_2 independent of z . Combining (6.6) and (6.7), we have

$$(6.8) \quad \|(-\Delta + 1 - z^2)^{-1} A_z f\|_{-s} \leq C_3 \|f\|_s \quad (f \in [\mathcal{S}(\mathbf{R}^3)]^4),$$

where C_3 is independent of $z \in J$.

In order to apply Lemma 5.7 to B_z , we note that

$$(6.9) \quad |\langle \xi \rangle^2 - z^2| \geq \frac{1}{2} \langle \xi \rangle$$

for $z \in J$ and $\xi \in \text{supp}[1 - \gamma_z]$. Using (6.9), we see that for any α there exists a constant C'_α such that

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left\{ \frac{1 - \gamma_z(\xi)}{\langle \xi \rangle^2 - z^2} \right\} \right| \leq C'_\alpha \langle \xi \rangle^{-1}$$

for all $z \in J$ and all $\xi \in \mathbf{R}^3$. Hence, to each α we can find a constant C''_α satisfying

$$\left| \left(\frac{\partial}{\partial \xi} \right)^\alpha \left[\frac{1 - \gamma_z(\xi)}{\langle \xi \rangle^2 - z^2} \widehat{L}_0(\xi) \right] \right| \leq C''_\alpha$$

for all $z \in J$ and all $\xi \in \mathbf{R}^3$. We then apply Lemma 5.7 to B_z , and deduce that

$$(6.10) \quad \|B_z f\|_{-s} \leq C_4 \|f\|_s \quad (f \in [\mathcal{S}(\mathbf{R}^3)]^4)$$

with a constant C_4 independent of $z \in J$. Since $[\mathcal{S}(\mathbf{R}^3)]^4$ is dense in $\mathcal{L}_{2,s}$, we conclude from (6.8), (6.10) and (6.7) that (6.3) holds. \square

§7. Proof of Theorem 2.5

In order to prove Theorem 2.5, we need some prerequisites and a few lemmas. Throughout this section, we regard $\mathcal{S}(\mathbf{R}^3)$ as a Fréchet space equipped with the semi-norms

$$(7.1) \quad |a|_{\ell, \mathcal{S}} = \sum_{|\alpha+\beta| \leq \ell} \sup_x \left\{ \left| x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta a(x) \right| \right\} \quad (\ell = 0, 1, 2, \dots).$$

For $f \in [\mathcal{S}(\mathbf{R}^3)]^4$ we introduce semi-norms by

$$|f|_{\ell, \mathcal{S}} = \sum_{k=1}^4 |f_k|_{\ell, \mathcal{S}} \quad (\ell = 0, 1, 2, \dots).$$

It is then trivial that $[\mathcal{S}(\mathbf{R}^3)]^4$ is a Fréchet space. Note that we use the same notation $|\cdot|_{\ell, \mathcal{S}}$ as in (7.1). We believe that this causes no confusion. For $f \in [\mathcal{S}(\mathbf{R}^3)]^4$ we define

$$\text{supp}[f] = \bigcup_{k=1}^4 \text{supp}[f_k].$$

Lemma 7.1.

Define

$$\mathcal{X}_0 = \left\{ f \in [\mathcal{S}(\mathbf{R}_x^3)]^4 / \mathcal{F}f \in [C_0^\infty(\mathbf{R}_\xi^3)]^4 \right\}.$$

Then \mathcal{X}_0 is dense in $\mathcal{L}_{2,s}$ for any $s \in \mathbf{R}$.

Proof. Let s be in \mathbf{R} . Let $g \in \mathcal{L}_{2,s}$ and $\epsilon > 0$ be given. Since $[\mathcal{S}(\mathbf{R}^3)]^4$ is dense in $\mathcal{L}_{2,s}$, we can find $f_\epsilon \in [\mathcal{S}(\mathbf{R}^3)]^4$ such that

$$(7.2) \quad \|g - f_\epsilon\|_s < \frac{\epsilon}{2}.$$

Note that $[C_0^\infty(\mathbf{R}^3)]^4$ is dense in $[\mathcal{S}(\mathbf{R}^3)]^4$. We then see that there exists a sequence $\{v_n\}_{n=1}^\infty \subset [C_0^\infty(\mathbf{R}_\xi^3)]^4$ such that

$$(7.3) \quad v_n \longrightarrow \mathcal{F}f_\epsilon \quad \text{in } [\mathcal{S}(\mathbf{R}_\xi^3)]^4 \quad \text{as } n \rightarrow \infty.$$

Now put

$$(7.4) \quad g_n = \mathcal{F}^{-1}v_n \quad (n = 1, 2, \dots).$$

Since \mathcal{F}^{-1} is a continuous map from $[\mathcal{S}(\mathbf{R}_\xi^3)]^4$ to $[\mathcal{S}(\mathbf{R}_x^3)]^4$, we deduce from (7.3) that $g_n \rightarrow f_\epsilon$ in $[\mathcal{S}(\mathbf{R}_x^3)]^4$ as $n \rightarrow \infty$. In particular, we have

$$\|g_n - f_\epsilon\|_s \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore we can choose an integer N so that

$$(7.5) \quad \|f_\epsilon - g_N\|_s < \frac{\epsilon}{2}.$$

Then we see, by (7.2) and (7.5), that

$$\|g - g_N\|_s < \epsilon,$$

and, by (7.4), that $g_N \in \mathcal{X}_0$. \square

Lemma 7.2.

For $z \in \mathbf{C}$, put

$$R(\xi; z) = \frac{1}{\langle \xi \rangle^2 - z^2} (\widehat{L}_0(\xi) + zI).$$

Then for any $K > 1$ and any multi-index α there exists a constant $C_{\alpha K} > 0$ such that

$$(7.6) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha R(\xi; z) \right| \leq \frac{C_{\alpha K}}{|z|}$$

for all ξ and z satisfying $\langle \xi \rangle \leq K$ and $|z| \geq 2K$.

Proof. We prove the lemma by induction on the length of α . Let $K > 1$. If $\langle \xi \rangle \leq K$ and $|z| \geq 2K$, then we see that

$$(7.7) \quad |\langle \xi \rangle^2 - z^2| \geq \frac{3}{4} |z|^2$$

and

$$|\widehat{L}_0(\xi) + zI| \leq C|z|.$$

Hence

$$(7.8) \quad |R(\xi; z)| \leq \frac{C}{|z|} \quad (\langle \xi \rangle \leq K, |z| \geq 2K),$$

which proves (7.6) for $\alpha = 0$. We next prove (7.6) for α with $|\alpha| = 1$. Since

$$\frac{\partial R}{\partial \xi_j} = \frac{1}{\langle \xi \rangle^2 - z^2} \left(\frac{\partial \widehat{L}_0}{\partial \xi_j} - 2R(\xi; z) \xi_j \right) \quad (j = 1, 2, 3),$$

we deduce from (7.7) and (7.8) that

$$(7.9) \quad \left| \frac{\partial R}{\partial \xi_j}(\xi; z) \right| \leq \frac{C}{|z|^2} \quad (\langle \xi \rangle \leq K, |z| \geq 2K).$$

Here we have used the fact that $\partial \widehat{L}_0 / \partial \xi_j$ is a constant matrix. It is evident that (7.9), in particular, proves (7.6) for α with $|\alpha| = 1$.

We now prove (7.6) for α with $|\alpha| \geq 2$. To this end, we differentiate the both sides of

$$(7.10) \quad (\langle \xi \rangle^2 - z^2) R(\xi; z) = \widehat{L}_0(\xi) + zI,$$

and apply the Leibniz formula to the product on the left hand side of (7.10). Then we get for α with $|\alpha| \geq 2$

$$(7.11) \quad \left(\frac{\partial}{\partial \xi} \right)^\alpha R = - \sum_{\substack{\beta \leq \alpha \\ |\beta|=1}} \binom{\alpha}{\beta} 2\xi^\beta \left(\frac{\partial}{\partial \xi} \right)^{\alpha-\beta} R - \sum_{\substack{\beta \leq \alpha \\ |\beta|=2}} \binom{\alpha}{\beta} 2\delta_\beta \left(\frac{\partial}{\partial \xi} \right)^{\alpha-\beta} R,$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

and $\delta_\beta = 1$ if β is one of the following indicies $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$ and $\delta_\beta = 0$ otherwise. It is clear that (7.11), together with (7.8) and (7.9), enables us to make an induction argument on $|\alpha|$. We omit the further details. \square

Lemma 7.3.

Let $R(\xi; z)$ be the same as in Lemma 7.2. Then for any $K > 1$ and any multi-index α there exists a constant $C_{\alpha K} > 0$ such that

$$(7.12) \quad \left| \left(\frac{\partial}{\partial \xi} \right)^\alpha [R(\xi; z_1) - R(\xi; z_2)] \right| \leq C_{\alpha K} |z_1 - z_2|$$

for all z_1, z_2 and ξ satisfying $|z_1|, |z_2| \geq 2K$ and $\langle \xi \rangle \leq K$.

Proof. Since $R(\xi; z) = (\widehat{L}_0(\xi) - z)^{-1}$, we see that

$$(7.13) \quad R(\xi; z_1) - R(\xi; z_2) = (z_1 - z_2) R(\xi; z_1) R(\xi; z_2).$$

Applying the Leibniz formula to the right hand side of (7.13) and using (7.6), we get (7.12) \square

Lemma 7.4.

Let $f \in [\mathcal{S}(\mathbf{R}^3)]^4$ satisfy

$$(7.14) \quad \text{supp}[\widehat{f}] \subset \{ \xi \in \mathbf{R}^3 / \langle \xi \rangle \leq K \}$$

for some $K > 1$. For each z with $|z| \geq 2K$, define

$$v_z(x) = \mathcal{F}^{-1} [R(\xi; z)] \mathcal{F} f,$$

where $R(\xi; z)$ is the same as in Lemma 7.2. Then

(i) For each $\ell \geq 0$, there corresponds a constant C_ℓ , depending also on f , such that

$$|v_z|_{\ell, \mathcal{S}} \leq \frac{C_\ell}{|z|}.$$

(ii) For any $\lambda \in (-\infty, -2K] \cup [2K, \infty)$,

$$v_{\lambda \pm i\mu} \rightarrow v_\lambda \quad \text{in} \quad [\mathcal{S}(\mathbf{R}^3)]^4 \quad \text{as} \quad \mu \downarrow 0.$$

Proof. Let α and β be multi-indices. By differentiation under the integral sign and integration by parts, we see that

$$(7.15) \quad x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta v_z(x) = (2\pi)^{-3} \int_{\mathbf{R}^3} e^{ix \cdot \xi} \left(i \frac{\partial}{\partial \xi} \right)^\alpha \{ R(\xi; z) (i\xi)^\beta \widehat{f}(\xi) \} d\xi.$$

By Lemma 7.2, we get

$$\begin{aligned} & | \text{the integrand on the RHS of (7.15)} | \\ & \leq C_{\alpha\beta K} \frac{1}{|z|} |\widehat{f}|_{|\alpha+\beta|+4, \mathcal{S}} \langle \xi \rangle^{-4}. \end{aligned}$$

Thus we obtain

$$|x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta v_z(x)| \leq \frac{C_{\alpha\beta K}}{|z|},$$

which implies conclusion (i). Similarly, using Lemma 7.3, we can show that

$$|x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta (v_{\lambda \pm i\mu}(x) - v_\lambda(x))| \leq C_{\alpha\beta K} \times \mu \quad (\mu > 0).$$

This leads us to conclusion (ii). \square

Proof of Theorem 2.5. In view of Theorem 2.4 and Lemma 7.1, it is sufficient to show that (2.4) is true for any $f \in \mathcal{X}_0$. Let f be in \mathcal{X}_0 and choose $K > 1$ so that (7.14) is valid. Define v_z in the same manner as in Lemma 7.4. Recalling (6.2), we remark that $R_0(z)f = v_z$ for z with $\text{Im } z \neq 0$. Then by Lemma 7.4(ii) and Theorem 2.2, together with Remark 2.3(ii), we see that

$$R_0^\pm(\lambda)f = v_\lambda \quad (\lambda \in (-\infty, -2K] \cup [2K, \infty)).$$

Moreover, by Lemma 7.4(i), we have

$$\|R_0^\pm(\lambda)f\|_{-s} \leq \frac{C_s}{|\lambda|},$$

which trivially implies (2.4). \square

§8. Proof of Theorem 2.6

Throughout this section we assume that $Q(x)$ satisfies Assumption 2.1.

Lemma 8.1.

Suppose that $1/2 < s < (1 + \epsilon)/2$. Then there exists a constant $C_ > 0$ such that*

$$(8.1) \quad \|Q R_0(z)f\|_s \leq C_* \|f\|_s$$

for all $f \in \mathcal{L}_{2,s}$ and all $z \in J$, where J is the set introduced in the beginning of the proof of Theorem 6.1.

Proof. Since $s - 1 - \epsilon < -s$, we have

$$(8.2) \quad \|Qf\|_s \leq C_1 \|f\|_{-s} \quad (f \in \mathcal{L}_{2,-s}),$$

where C_1 is a constant depending only on the constant K appearing in the Assumption 2.1. Combining (8.2) with Theorem 6.1 gives the lemma. \square

Writing

$$R_t(z) = (H_t - z)^{-1}$$

we see that

$$(8.3) \quad R_t(z) (I + tQ R_0(z)) = R_0(z) \quad \text{on } \mathcal{L}_2.$$

According to Lemma 8.1, we can regard $I + tQ R_0(z)$ as a bounded operator in $\mathcal{L}_{2,s}$ provided that $1/2 < s < (1 + \epsilon)/2$. Letting C_* be the constant in (8.1) and choosing $t_0 > 0$ so that

$$(8.4) \quad 0 < t_0 C_* < 1,$$

we can construct the inverse of $I + tQ R_0(z)$ by using the Neumann series in $\mathbf{B}(\mathcal{L}_{2,s}, \mathcal{L}_{2,s})$ if $|t| \leq t_0$ and $z \in J$:

$$(8.5) \quad (I + tQ R_0(z))^{-1} = \sum_{\ell=0}^{\infty} (-tQ R_0(z))^{\ell}.$$

Proof of Theorem 2.6(i). We may assume, without loss of generality, that $1/2 < s < (1 + \epsilon)/2$. Indeed, if $1/2 < s < s'$, then

$$\|R_t^{\pm}(\lambda)\|_{(s', -s')} \leq \|R_t^{\pm}(\lambda)\|_{(s, -s)}.$$

In view of Theorem 2.2, it is sufficient to show that there exists a constant $C_2 > 0$ such that

$$(8.6) \quad \|R_t(z)\|_{(s, -s)} \leq C_2$$

for all t with $|t| \leq t_0$ and all $z \in J$. By (8.5) and Lemma 8.1, we see that

$$(8.7) \quad \|(I + tQ R_0(z))^{-1}\|_{(s,s)} \leq \frac{1}{1 - t_0 C_*} \quad (|t| \leq t_0, \ z \in J).$$

Hence, using (8.3) and (8.7), we have

$$\|R_t(z)\|_{(s,-s)} \leq \|R_0(z)\|_{(s,-s)} \frac{1}{1 - t_0 C_*}$$

when $|t| \leq t_0$ and $z \in J$. Combining this inequality with Theorem 6.1, we obtain (8.6). \square

Proof of Theorem 2.6(ii). We may assume again, without loss of generality, that $1/2 < s < (1 + \epsilon)/2$. In fact, if $1/2 < s < s'$, then $\|f\|_{-s'} \leq \|f\|_{-s}$.

Suppose that $f \in \mathcal{L}_{2,s}$, $z \in J$ and $|t| \leq t_0$. Then, by (8.3) and (8.5), we have

$$(8.8) \quad \begin{aligned} \|R_t(z)f\|_{-s} &\leq \|R_0(z)f\|_{-s} + \|R_0(z) \sum_{\ell=1}^N (-t Q R_0(z))^\ell f\|_{-s} \\ &\quad + \|R_0(z) \sum_{\ell=N+1}^{\infty} (-t Q R_0(z))^\ell f\|_{-s} \end{aligned}$$

for any positive integer N . The second term on the right hand side of (8.8) is estimated by

$$\begin{aligned} &\|R_0(z)\|_{(s,-s)} \left\{ \sum_{\ell=1}^N \|(-t Q R_0(z))^{\ell-1}\|_{(s,s)} \right\} |t| \|Q R_0(z)f\|_s \\ &\leq C_3 \left\{ \sum_{\ell=1}^N (t_0 C_*)^{\ell-1} \right\} t_0 C_1 \|R_0(z)f\|_{-s} \end{aligned}$$

where C_1 is the same constant as in (8.2) and C_3 is the constant given by the supremum in Theorem 6.1. With this notation, the last term on the right hand side of (8.8) is less than or equal to

$$C_3 \left\{ \sum_{\ell=N+1}^{\infty} (t_0 C_*)^\ell \right\} \|f\|_s.$$

Summing up, we get

$$(8.9) \quad \begin{aligned} \|R_t(z)f\|_{-s} &\leq \left\{ 1 + t_0 C_1 C_3 \sum_{\ell=1}^N (t_0 C_*)^{\ell-1} \right\} \|R_0(z)f\|_{-s} \\ &\quad + C_3 \left\{ \sum_{\ell=N+1}^{\infty} (t_0 C_*)^\ell \right\} \|f\|_s \end{aligned}$$

for any positive integer N . We now replace z in (8.9) with $\lambda \pm i\mu$ ($|\lambda| > 2$, $0 < \mu < 1$) and take the limits as $\mu \downarrow 0$. Then we obtain, by Theorem 2.2,

$$(8.10) \quad \begin{aligned} \|R_t^\pm(\lambda) f\|_{-s} &\leq \left\{ 1 + t_0 C_1 C_3 \sum_{\ell=1}^N (t_0 C_*)^{\ell-1} \right\} \|R_0^\pm(\lambda) f\|_{-s} \\ &\quad + C_3 \left\{ \sum_{\ell=N+1}^{\infty} (t_0 C_*)^\ell \right\} \|f\|_s \end{aligned}$$

for any positive integer N . Theorem 2.5, together with (8.10), implies that

$$\limsup_{|\lambda| \rightarrow \infty} \|R_t^\pm(\lambda) f\|_{-s} \leq C_3 \left\{ \sum_{\ell=N+1}^{\infty} (t_0 C_*)^\ell \right\} \|f\|_s.$$

Since N is arbitrary and $0 < t_0 C_* < 1$ (recall (8.4)), we conclude that

$$\lim_{|\lambda| \rightarrow \infty} \|R_t^\pm(\lambda) f\|_{-s} = 0$$

for $f \in \mathcal{L}_{2,s}$ and t with $|t| \leq t_0$. \square

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